Int. J. Solids Structures Vol. 21, No. 5, pp. 489–501, 1985
Printed in the U.S.A.

0020-7683/85 $3.00 + .00

INTERACTIONS BETWEEN A CURVED CRACK AND A CIRCULAR BOUNDARY

CHANG SHANGCHOW
Department of Aircraft Engineering, Northwestern Polytechnical University, Xian, Shaanxi, People's Republic of China

(*Received 25 January* 1983; *in revised form 23 May* 1984)

Abstract—Taking the stress intensity factor at crack tips as the predominant unknown, the present article solves two kinds of plane interaction problems between a curved crack and a circular boundary with use of a special form of the alternating method in which the key roles in the alternating process are played by some simple coefficients. Based on the exact formulae for the coefficients, equivalent procedures are developed that reduce the original interaction problems between a curved crack and a circular boundary to plane problems of a single curved crack in an infinite plate. Making use of the equivalent procedures, accurate values of stress intensity factors are obtained readily and effectively.

1. INTRODUCTION

Stress analysis for cracked discs subjected to plane loading is of practical significance in engineering. Nevertheless, treatment of the subject in the published literature appears to be confined to discs with flat cracks, and no research reference seems available in situations where the cracks contained in discs are curved ones[1–3].

A similar state of affairs can be found for cases of a crack interacting with a neighbouring circular hole contained in an infinite plate. The existing solutions for this problem again seem limited to flat cracks, while the corresponding solutions for curved ones still remain to be established.

The two problems described above, or more distinctly the stress intensity factor evaluation problems for a disc containing a concentric arc crack and for an infinite plate with a circular hole and a concentric arc crack, are a pair of particular aspects of the general subject of plane interactions between a curved crack and a circular boundary and are to be treated in this study by use of a special form of the alternating method. The key roles throughout the alternating cycles described in the following sections are played by some simple coefficients. The exact formulae for the coefficients developed in the next sections then enable us to establish equivalent procedures that reduce the original plane interaction problems between a curved crack and a circular boundary exactly to the plane problems of a single curved crack in an infinite plate. Making use of the equivalent procedures, the alternating cycles can be carried on, and the values of stress intensity factors obtained systematically and succinctly.

A number of numerical results of stress intensity factors of a curved crack within or outside a circular boundary are presented in the last section to provide some useful data in engineering and demonstrate the validity and efficiency of the equivalent procedures.

2. STRESS BOUNDARY PROBLEM OF A SINGLE CURVED CRACK IN AN INFINITE PLATE

For the purpose of solving the interaction problems between a curved crack and a circular boundary, the following three fundamental solutions must be prepared before the beginning of the alternating cycles: (i) the stress boundary problem of a solid disc; (ii) the stress boundary problem of an infinite plate containing a circular hole; (iii) the stress boundary problem of an infinite plate containing a curved crack. The first two solutions are available and the third one can be developed by the method of Muskhelishvili[4].

Consider the infinite plate containing a single circular arc crack as illustrated in Fig. 1. It is intended to solve the plane problem of elasticity for the aforementioned plate governed by the following conditions

$$\text{at infinity,} \quad \sigma_r = \sigma_\theta = \tau_{r\theta} = 0 \tag{1}$$

$$\text{on arc } L \text{ (the crack),} \quad \begin{cases} \sigma_r^+ + \sigma_r^- + i(\tau_{r\theta}^+ + \tau_{r\theta}^-) = 2p(t) & (2) \\ \sigma_r^+ - \sigma_r^- + i(\tau_{r\theta}^+ - \tau_{r\theta}^-) = 0, & (3) \end{cases}$$

where t denotes the z coordinate on L and $p(t)$ stands for the external loading function. In the complex variable approach, the stresses in the plate can be expressed by two fundamental functions, such as

$$\sigma_r + \sigma_\theta = 2[\Phi(z) + \overline{\Phi(z)}] \tag{4}$$

$$\sigma_r + i\tau_{r\theta} = \Phi(z) + \Omega\left(\frac{1}{\bar{z}}\right) + \bar{z}\left(\bar{z} - \frac{1}{z}\right)\Psi(z), \tag{5}$$

where

$$\Psi(z) = \frac{1}{z^2}\Phi(z) - \frac{1}{z^2}\bar{\Omega}(z) - \frac{1}{z}\Phi'(z). \tag{6}$$

Utilising eqns (4) and (5), eqns (2) and (3) can be written as

$$[\Phi(t) + \Omega(t)]^+ + [\Phi(t) + \Omega(t)]^- = 2p(t) \tag{7}$$

$$[\Phi(t) - \Omega(t)]^+ - [\Phi(t) - \Omega(t)]^- = 0. \tag{8}$$

The solution of the stress boundary problem is then reduced to finding two functions, $\Phi(z)$ and $\Omega(z)$, holomorphic in the complete plane cut by arc L (the crack) and taking constant values at infinity. The solutions for this reduced problem are available and take the forms of

$$\Phi(z) = \frac{1}{2\pi i X(z)}\int_L \frac{X^+(t)p(t)}{t - z}\,dt + \frac{1}{2X(z)}\left(C_0 z + C_1 + \frac{D_1}{z} + \frac{D_2}{z^2}\right) + \frac{D_0}{2} \tag{9}$$

$$\Omega(z) = \Phi(z) - D_0, \tag{10}$$

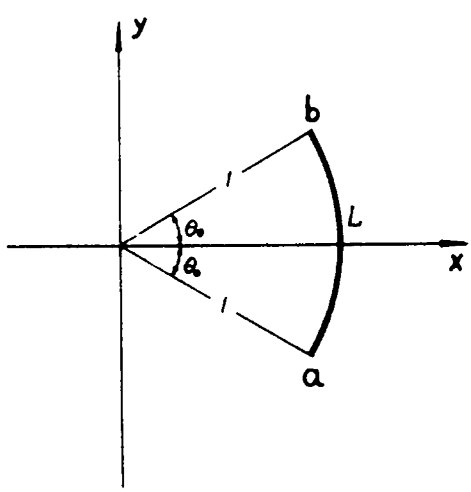

Fig. 1. The crack geometry.

where $X(z)$ denotes a branch of the multivalued function $\sqrt{(z-a)(z-b)} = \sqrt{z^2 - 2\cos\theta_0 z + 1}$, holomorphic in the complete plane cut by L and satisfying the following conditions:

$$\lim_{z\to\infty} \frac{X(z)}{z} = 1, \qquad \lim_{z\to 0} X(z) = -1.$$

Therefore, $1/X(z)$ and $X(z)$ can be put, respectively, into the following series forms:

$$\frac{1}{X(z)} = \begin{cases} \dfrac{1}{z}\sum_{0}^{\infty} P_n \left(\dfrac{1}{z}\right)^n, & |z| > 1 \quad (11) \\[2ex] -\sum_{0}^{\infty} P_n z^n, & |z| < 1 \quad (12) \end{cases}$$

$$X(z) = \begin{cases} z\sum_{0}^{\infty} Q_n \left(\dfrac{1}{z}\right)^n, & |z| > 1 \quad (13) \\[2ex] -\sum_{0}^{\infty} Q_n z^n, & |z| < 1, \quad (14) \end{cases}$$

where $P_n \equiv P_n(\cos\theta_0)$ represents the Legendre polynomials and the polynomials Q_n are defined by

$$Q_n = P_n - 2\cos\theta_0 P_{n-1} + P_{n-2}, \qquad n = 0, 1, 2, \ldots$$

with $P_{-1} = P_{-2} = 0$.

In the evaluation of the integral appearing in eqn (9), we can express the function $p(t)$ in the following general form with use of a power series of t such that

$$p(t) = \sum_{-\infty}^{\infty} a_n t^n. \tag{15}$$

For the sake of simplicity, the values of a_n are regarded as real number (corresponding to the loading conditions symmetric to the real axis). But situations in which a_n values are imaginary can be treated in a closely similar manner.

Substituting $p(t) = t^m$ $(m = 0, 1, 2, \ldots; -1, -2, \ldots)$ successively in the integral term of eqn (9) and carrying out some rather cumbersome manipulations, the following result is obtained:

$$\int_L \frac{X^+(t)p(t)}{t-z}\,dt = \begin{cases} \pi i\left[X(z)z^m - \sum_{0}^{m+1} Q_n z^{m+1-n}\right], & m = 0, 1, 2, \ldots \quad (16) \\[2ex] \pi i\left[\dfrac{X(z)}{z} - 1 + \dfrac{1}{z}\right], & m = -1 \quad (17) \\[2ex] \pi i\left[X(z)z^m + \sum_{0}^{|m|-1} Q_n z^{m+n}\right], & m = -2, -3, \ldots \quad (18) \end{cases}$$

Now consider the constants C_0, C_1, D_0, D_1 and D_2 contained in formula (9). Since the stresses at infinity and the resultant of the external loads acting on the crack surfaces vanish, $D_1 = D_2 = 0$. The remaining three constants can be determined from a procedure described in [4] in a straightforward manner. The final results are

$$C_0^m = -\frac{C_1^m}{\cos\theta_0} = -D_0^m = \begin{cases} \dfrac{1-\cos\theta_0}{3-\cos\theta_0}, & m = 0, -1 \qquad (19) \\[2ex] \dfrac{Q_{m-1}}{3-\cos\theta_0}, & m = 1, 2, 3, \ldots \qquad (20) \\[2ex] -\dfrac{\sum\limits_{1}^{|m|} P_k Q_{|m|-k}}{3-\cos\theta_0}, & m = -2, -3, \ldots \qquad (21) \end{cases}$$

In the above formulae, as well as in the following, the superscript (or subscript) m is used to denote that the corresponding constant (or function) is related to $p(t) = t^m$. Synthesising the results obtained leads to the following expressions for $\Phi_m(z)$:

$$\Phi_m(z) = \begin{cases} \dfrac{1}{2}\left[z^m - \dfrac{\sum\limits_0^{m+1} Q_n z^{m+1-n}}{X(z)}\right] + \dfrac{C_0^m z + C_1^m}{2X(z)} + \dfrac{D_0^m}{2}, & m = 0, 1, 2, \ldots \qquad (22) \\[3ex] \dfrac{1}{2}\left[\dfrac{1}{z} - \dfrac{1}{X(z)} + \dfrac{1}{zX(z)}\right] + \dfrac{C_0^{-1} z + C_1^{-1}}{2X(z)} + \dfrac{D_0^{-1}}{2}, & m = -1 \qquad (23) \\[3ex] \dfrac{1}{2}\left[z^m + \dfrac{\sum\limits_0^{|m|-1} Q_n z^{m+n}}{X(z)}\right] + \dfrac{C_0^m z + C_1^m}{2X(z)} + \dfrac{D_0^m}{2}, & m = -2, -3, \ldots \qquad (24) \end{cases}$$

By use of eqn (11), the following series forms of $\Phi_m(z)$ can be obtained for $|z| > 1$ ($m = 0, \pm 1, \pm 2, \ldots$):

$$\Phi_m(z) = \frac{1}{2}\sum_1^\infty (S_n^m + C_0^m P_n + C_1^m P_{n-1})\left(\frac{1}{z}\right)^n \equiv \frac{1}{2}\sum_1^\infty B_n^m \left(\frac{1}{z}\right)^n, \qquad (25)$$

where

$$S_n^m = \begin{cases} -\sum\limits_{k=0}^{m+1} Q_k P_{n+m-k}, & m = 0, 1, 2, \ldots \\ P_{n-2} - I(n-1)P_{n-1}, & m = -1 \\ \sum\limits_{k=0}^{|m|-1} I(n+k-|m|)Q_k P_{n+k-|m|-1} + I(|m|-n+1)I(n-|m|+1), & m = -2, -3, \ldots \end{cases}$$

In the above formulae, $I(n)$ denotes a function of integer n and is defined by

$$I(n) = \frac{\operatorname{sgn}(n) + \operatorname{sgn}(|n|)}{2}.$$

A similar procedure can be applied to $\Phi_m(z)$ for $|z| < 1$ ($m = 0, \pm 1, \pm 2, \ldots$)

to obtain

$$\Phi_m(z) = \frac{1}{2}\sum_0^{\infty} [T_n^m - C_0^m P_{n-1} - C_1^m P_n + I(1-n)D_0^m]z^n \equiv \frac{1}{2}\sum_0^{\infty} E_n^m z^n, \qquad (26)$$

where

$$T_n^m = \begin{cases} \sum_{k=0}^{m+1} Q_k P_{n-m-1+k} + I(n-m+1)I(m-n+1), & m = 0, 1, 2, \ldots \\ P_n - P_{n+1}, & m = -1 \\ -\sum_{k=0}^{|m|-1} Q_k P_{n+|m|-k}, & m = -2, -3, \ldots \end{cases}$$

Substituting eqns (10), (25) and (26) into eqn (6), the following expansions for $\Psi_m(z)$ are obtained:

$$\Psi_m(z) = \frac{1}{2}\sum_1^{\infty} \{(n-1)B_{n-2}^m - I(n-1)[E_{n-2}^m - 2D_0^m I(3-n)]\}\left(\frac{1}{z}\right)^n$$

$$\equiv \frac{1}{2}\sum_1^{\infty} F_n^m \left(\frac{1}{z}\right)^n, \quad |z| > 1; \quad m = 0, \pm 1, \pm 2, \ldots \qquad (27)$$

$$\Psi_m(z) = \frac{1}{2}\sum_0^{\infty} (-(1+n)E_{n+2}^m - B_{n+2}^m]z^n$$

$$\equiv \frac{1}{2}\sum_0^{\infty} G_n^m z^n, \quad |z| < 1; \quad m = 0, \pm 1, \pm 2, \ldots \qquad (28)$$

with $B_0^m = B_{-1}^m = E_{-1}^m = 0$.

Making use of the expressions for $\Phi_m(z)$ and $\Psi_m(z)$ derived above, the stresses in the cracked plate can be determined. Among them, from the available formula for stress intensity factors of a curved crack (see Fig. 2 for the meanings of z_0 and α),

$$K = K_{\rm I} - iK_{\rm II} = \lim_{z \to z_0} 2\sqrt{2e^{-i\alpha}(z - z_0)}\,\Phi(z),$$

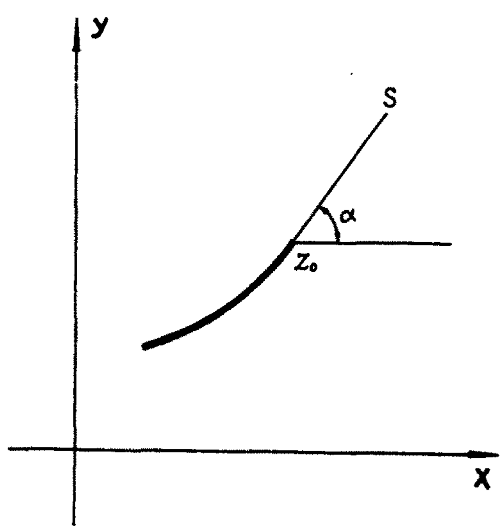

Fig. 2. The crack tip.

the following stress intensity factor evaluation formulae can be deduced:

$$K_b^m = K_{Ib}^m - iK_{IIb}^m = -\frac{ie^{-i\theta_0/2}\bar{\Phi}_m(b)}{\sqrt{\sin\theta_0}} \tag{29}$$

$$K_a^m = K_{Ia}^m - iK_{IIa}^m = \frac{ie^{i\theta_0/2}\bar{\Phi}_m(a)}{\sqrt{\sin\theta_0}}, \tag{30}$$

where

$$\bar{\Phi}_m(z) = \begin{cases} -\sum_0^{m+1} Q_n z^{m+1-n} + C_0^m z + C_1^m, & m = 0, 1, 2, \ldots \quad (31) \\ -1 + \frac{1}{z} + C_0^{-1}z + C_1^{-1}, & m = -1 \quad (32) \\ \sum_0^{|m|-1} Q_n z^{m+n} + C_0^m z + C_1^m, & m = -2, -3, \ldots \quad (33) \end{cases}$$

For the case of $p(t) = -1$, eqns (31) and (32) give

$$(K_b, K_a) = -(K_b^0, K_a^0) = \frac{\sqrt{\sin\theta_0}}{1 + \sin^2(\theta_0/2)}(e^{-i\theta_0/2}, e^{i\theta_0/2}),$$

and for the case of $p(t) = (\frac{1}{2})(t^{-2} - 1)$ (this case is equivalent to the situations in which the plate is under the action of uniform tensive stress at infinity in the direction of the y-axis),

$$K_b = \frac{1}{2}(-K_b^0 + K_b^{-2}) = \frac{\sqrt{\sin\theta_0}}{2}\left[\frac{1 + \sin^2\theta_0/2\cos^2\theta_0/2}{1 + \sin^2\theta_0/2}e^{-i\theta_0/2} + e^{-3\theta_0 i/2}\right]$$

$$K_a = \frac{1}{2}(-K_a^0 + K_a^{-2}) = \frac{\sqrt{\sin\theta_0}}{2}\left[\frac{1 + \sin^2\theta_0/2\cos^2\theta_0/2}{1 + \sin^2\theta_0/2}e^{i\theta_0/2} - e^{3\theta_0 i/2}\right].$$

The above results agree perfectly with the available ones[1].

Because $a = \bar{b}$, it can be seen from eqns (29) and (30) that K_a and K_b are composed of conjugate complex numbers.

3. THE STRESS BOUNDARY PROBLEM OF A DISC WITH A CURVED CRACK AND AN EQUIVALENT PROCEDURE

Consider the circular disc containing a curved (circumferential) crack, as shown in Fig. 3, with the circular boundary and the crack defined by $z = Re^{i\theta}(R > 1, -\pi < \theta \le \pi)$ and $z = e^{i\theta}(-\theta_0 \le \theta \le \theta_0)$, respectively. It is desired to determine the stress intensity factors of the cracked disc, provided the disc is subject to arbitrary plane loads on the circumference.

For the mere purpose of determining stress intensity factors, by the use of a simple argument of superposition well known in fracture mechanics, it can be easily seen that the stress boundary conditions in the problem described above can be replaced by

$$\text{on the circumference,} \quad \sigma_r = \tau_{r\theta} = 0 \tag{34}$$

$$\text{on the crack surfaces,} \quad \begin{cases} \sigma_r^+ + \sigma_r^- + i(\tau_{r\theta}^+ + \tau_{r\theta}^-) = 2p(t) \\ \sigma_r^+ - \sigma_r^- + i(\tau_{r\theta}^+ - \tau_{r\theta}^-) = 0. \end{cases} \tag{35}$$

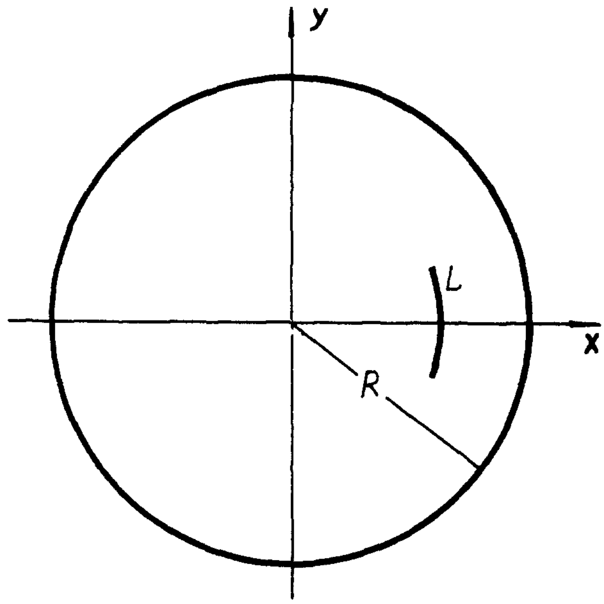

Fig. 3. The cracked disc.

Again, without the loss of generality, $p(t)$ can be expressed by a power series in t as indicated in eqn (15). Therefore, it is necessary only to consider the typical case of $p(t) = t^m$. The general solution can be composed by superposition.

The reduced problem characterised by eqns (34) and (35) is to be solved by the alternating method. First, it is noted that instead of satisfying the boundary condition (34), the stresses represented by the two fundamental stress functions $\Phi_m(z)$ and $\Psi_m(z)$ take the following form on $z = Re^{i\theta}$, the circumference[4]:

$$\sigma_{rm} - i\tau_{r\theta m} = \Phi_m(z) + \overline{\Phi_m(z)} - e^{2i\theta}[\bar{z}\Phi'_m(z) + \Psi_m(z)]. \tag{36}$$

From substituting eqns (25) and (27) into the above formula, the following equation results:

$$\sigma_{rm} - i\tau_{r\theta m} = \frac{1}{2}\left\{-\frac{F_2^m}{R^2} - \frac{F_1^m}{R}e^{i\theta} + \sum_1^\infty \frac{B_n^m}{R^n}e^{in\theta} + \sum_1^\infty\left[\frac{(n+1)B_n^m}{R^n} - \frac{F_{n+2}^m}{R^{n+2}}\right]e^{-in\theta}\right\} \equiv \sum_{-\infty}^{\infty} A_n^m e^{in\theta}, \qquad m = 0, \pm1, \pm2, \ldots \tag{37}$$

where

$$A_0^m = -\frac{F_2^m}{2R^2}, \qquad A_1^m = 0, \qquad A_n^m = \frac{B_n^m}{2R^n}, \qquad n = 2, 3, 4, \ldots$$
$$A_n^m = \frac{(|n|+1)B_{|n|}^m}{2R^{|n|}} - \frac{F_{|n|+2}^m}{2R^{|n|+2}}, \qquad n = -1, -2, -3, \ldots \tag{38}$$

On the other hand, the external load exerted on the circumference of the disc represented by eqn (36) strains the corresponding solid disc ($|z| \leq R$) in plane with the stresses, which can be expressed by the following two fundamental stress functions[4]:

$$\Phi_{(m)}(z) = \sum_0^\infty a_n^m z^n, \qquad \Psi_{(m)}(z) = \sum_0^\infty a_n'^m z^n. \tag{39}$$

Substituting eqn (39) into (36) and taking notice of eqns (37) and (38), a_n^m and $a_n'^m$ can be determined as

$$a_0^m = \frac{A_0^m}{2}, \qquad a_n^m = \frac{A_{-n}^m}{R_n}, \qquad n = 1, 2, 3, \ldots$$
$$a_n'^m = -\frac{1}{R^n}[(1 + n)A^m_{-(n+2)} + A^m_{n+2}], \qquad n = 0, 1, 2, \ldots \tag{40}$$

And the stresses on $L(z = e^{i\theta}, |\theta| \leq \theta_0)$ corresponding to $\Phi_{(m)}(z)$ and $\Psi_{(m)}(z)$ turn out to be

$$\begin{aligned}\sigma_r + i\tau_{r\theta} &= \Phi_{(m)}(t) + \overline{\Phi_{(m)}(t)} - e^{-2i\theta}[t\overline{\Phi'_{(m)}(t)} + \overline{\Psi_{(m)}(t)}] \\ &= \sum_0^\infty (a_n^m + I(1 - n)a_n^m)t^n + \sum_1^\infty [(1 - n)a_n^m - a_{n-2}'^m]t^{-n} \\ &\equiv \sum_{-\infty}^\infty W_n^m t^n\end{aligned} \tag{41}$$

with $a_{-1}'^m = 0$ and

$$W_0^m = -\frac{F_2^m}{2R^2}, \quad W_n^m = \frac{1}{2R^{2n}}\left[(n + 1)B_n^m - \frac{F_{n+2}^m}{R^2}\right], \qquad n = 1, 2, 3, \ldots$$
$$W_{-1}^m = 0, \qquad W_n^m = \left(\frac{1 - |n|}{2R^{2|n|}} + \frac{|n| - 1}{2R^{2|n|-2}}\right)\left[(1 + |n|)B_n^m - \frac{F_{|n|+2}^m}{R^2}\right] + \frac{B_{|n|}^m}{2R^{2|n|-2}},$$
$$n = -2, -3, \ldots \tag{42}$$

A straightforward analysis of the result obtained above shows that if in the beginning of an alternating cycle in the alternating procedure, the stresses applied on the crack surfaces are expressed by $\sigma_r + i\tau_{r\theta} = t^m$, then in the beginning of the next cycle, the stresses on the crack surfaces should be expressed exactly by eqn (41). To be more precise and general, in the case in which the N-th cycle in the alternating process is started with $\sigma_r + i\tau_{r\theta} = \sum_{-\infty}^{\infty} f_n t^n$ applied on the crack surfaces, then in the beginning of the $(N + 1)$-th cycle, the stresses applied on the crack surfaces should be expressed by $\sigma_r + i\tau_{r\theta} = \sum_{-\infty}^{\infty}(\sum_{k=-\infty}^{\infty} f_k W_n^k)t^n$. The above argument then leads to the following equivalent procedure, which reduces the original plane interaction problems between a curved crack and a circular boundary to the stress boundary problems of a single curved crack contained in an infinite plate.

3.1 *The equivalent procedure*

The stress intensity factors K_a and K_b in a circular disc ($|z| \leq R$) containing a circumferential crack ($z = e^{i\theta}$, $-\theta_0 \leq \theta \leq \theta_0$) (Fig. 3) with the crack surfaces loaded

$$\sigma_r + i\tau_{r\theta} = p(t) = \sum_{-\infty}^{\infty} a_n t^n \tag{43}$$

are equal to, respectively, the stress intensity factors K_a and K_b in an infinite plate containing a curved crack ($z = e^{i\theta}$, $-\theta_0 \leq \theta \leq \theta_0$) (Fig. 1) with the crack surfaces loaded

$$\sigma_r + i\tau_{r\theta} = \sum_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty} H_k^n\right)t^n \tag{44}$$

where

$$H_0^n = a_n, \qquad H_1^n = \sum_{k_1=-\infty}^{\infty} a_{k_1} W_n^{k_1}, \quad H_{N+1}^n = \sum_{k_1=-\infty}^{\infty} H_N^{k_1} W_n^{k_1}, \qquad N = 1, 2, 3, \ldots \tag{45}$$

It is obvious that with the aid of the equivalent procedure, the alternating cycles in the alternating process can be carried out systematically and succinctly to obtain accurate values of the stress intensity factors.

4. THE STRESS BOUNDARY PROBLEM OF AN INFINITE PLATE CONTAINING A CIRCULAR HOLE AND A CURVED CRACK AND AN EQUIVALENT PROCEDURE

Consider the infinite plate containing a circular hole ($|z| = R, R < 1$) and a curved crack ($z = e^{i\theta}$, $-\theta_0 \leq \theta \leq \theta_0$) as shown in Fig. 4. Applying an argument similar to that described in the foregoing section for the purpose of determining stress intensity factors, the original stress boundary conditions of the perforated and cracked plate can be reduced to

$$\text{at infinity,} \qquad \sigma_r = \sigma_\theta = \tau_{r\theta} = 0 \tag{46}$$

$$\text{on } |z| = R, \qquad \sigma_r = \tau_{r\theta} = 0 \tag{47}$$

$$\text{on } L(z = e^{i\theta},\ |\theta| \leq \theta_0) \quad \begin{cases} \sigma_r^+ + \sigma_r^- + i(\tau_{r\theta}^+ + \tau_{r\theta}^-) = 2p(t) \\ \sigma_r^+ - \sigma_r^- + i(\tau_{r\theta}^+ - \tau_{r\theta}^-) = 0. \end{cases} \tag{48}$$

As above, consider the typical case of $p(t) = t^m$. It is evident that instead of satisfying the boundary condition (47), the stresses represented by $\Phi_m(z)$ and $\Psi_m(z)$ strain the surface of the hole with

$$\begin{aligned} \sigma_r - i\tau_{r\theta} &= \Phi_m(z) + \overline{\Phi_m(z)} - e^{2i\theta}[\bar{z}\Phi_m'(z) + \Psi_m(z)]\,|_{z = Re^{i\theta}} \\ &= \frac{1}{2}\left\{\sum_0^\infty [(1-n)E_n^m R^n + I(1-n)E_n^m R^n - G_{n-2}^m R^{n-2}]e^{in\theta}\right. \\ &\quad + \sum_1^\infty E_n^m R^n e^{-in\theta} \equiv \sum_{-\infty}^{\infty} \bar{\bar{A}}_n^m e^{in\theta}. \end{aligned} \tag{49}$$

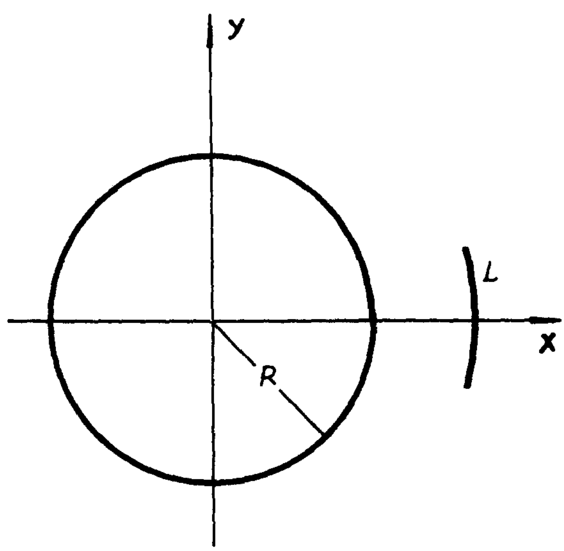

Fig. 4. The infinite plate with a circular hole and a curved crack.

In the above formula, $G^m_{-1} = G^m_{-2} = 0$ and

$$\bar{\bar{A}}^m_0 = E^m_0, \quad \bar{\bar{A}}^m_1 = 0, \quad \bar{\bar{A}}^m_n = \frac{(1-n)E^m_n R^n - G^m_{n-2}R^{n-2}}{2}, \quad n = 2, 3, \ldots$$

$$\bar{\bar{A}}^m_n = \frac{E^m_{|n|}R^{|n|}}{2}, \quad n = -1, -2, -3, \ldots \tag{50}$$

Denoting $\Phi_{(m)}(z)$, $\Psi_{(m)}(z)$ as two fundamental stress functions for the corresponding infinite plate containing a single circular hole and loaded along the hole boundary by the tractions expressed in eqn (49), the following formulae can be obtained:

$$\Phi_{(m)}(z) = \sum_0^\infty \bar{a}^m_n z^{-n}, \qquad \Psi_{(m)}(z) = \sum_0^\infty \bar{a}'^m_n z^{-n}. \tag{51}$$

The coefficients $\bar{a}^m_n$ and $\bar{a}'^m_n$ are given by

$$\bar{a}^m_0 = \bar{a}^m_1 = 0, \qquad \bar{a}^m_n = \bar{\bar{A}}^m_n R^n, \qquad n = 2, 3, \ldots$$

$$\bar{a}'^m_0 = \bar{a}^m_1 = 0, \qquad \bar{a}'^m_2 = -\bar{\bar{A}}^m_0 R^2, \qquad \bar{a}'^m_n = (n-1)\bar{a}^m_{n-2}R^2 - \bar{\bar{A}}^m_{2-n}R^n,$$

$$n = 3, 4, \ldots \tag{52}$$

The stresses on L (the crack) corresponding to functions $\Phi_{(m)}(z)$ and $\Psi_{(m)}(z)$ turn out to be

$$\sigma_r + i\tau_{r\theta} = -\bar{a}'^m_2 - \bar{a}'^m_3 e^{i\theta} + \sum_2^\infty [(1+n)\bar{a}^m_n - \bar{a}'^m_{n+2}]e^{in\theta} + \sum_2^\infty \bar{\bar{a}}^m_n e^{in\theta} \equiv \sum_{-\infty}^\infty \bar{\bar{W}}^m_n t^n, \tag{53}$$

where

$$\bar{\bar{W}}^m_0 = E^m_0 R^2, \quad \bar{\bar{W}}^m_1 = \frac{E^m_1 R^4}{2}$$

$$\bar{\bar{W}}^m_n = \{(1+n)[(1-n)E^m_n R^2 - G^m_{n-2}](1-R^2) + E^m_n R^2\}\frac{R^{2n-2}}{2},$$

$$n = 2, 3, \ldots \tag{54}$$

$$\bar{\bar{W}}^m_{-1} = 0, \quad \bar{\bar{W}}^m_n = [(1-|n|)E^m_{|n|}R^2 - G^m_{|n|-2}]\frac{R^{2|n|-2}}{2}, \qquad n = -2, -3, \ldots$$

Synthesizing the above results and using an argument similar to that described above, the equivalent procedure, which reduces the original plane interaction problem between a curved crack and a circular hole contained in an infinite plate to the stress boundary problems of a single curved crack in an infinite plate, is obtained.

4.1 *The equivalent procedure*

The stress intensity factors K_a and K_b in an infinite plate containing a circular hole ($|z| = R$) and a circumferential crack ($z = e^{i\theta}$, $-\theta_0 \leq \theta \leq \theta_0$) (Fig. 4) with the crack surfaces loaded by

$$\sigma_r + i\tau_{r\theta} = p(t) = \sum_{-\infty}^\infty a_n t^n \tag{55}$$

are equal to, respectively, the stress intensity factors K_a and K_b in an infinite plate containing a curved crack ($z = e^{i\theta}$, $-\theta_0 \leqslant \theta \leqslant \theta_0$) with the crack surfaces loaded by

$$\sigma_r + i\tau_{r\theta} = \sum_{-\infty}^{\infty} \left(\sum_{j=0}^{\infty} \overline{H}_j^n \right) t^n, \tag{56}$$

where

$$\overline{H}_0^n = a_n, \quad \overline{H}_1^n = \sum_{k_1=-\infty}^{\infty} a_{k_1} \overline{\overline{W}}{}_n^{k_1}, \quad \overline{H}_{N+1}^n = \sum_{k_1=-\infty}^{\infty} \overline{H}_N^{k_1} \overline{\overline{W}}{}_n^{k_1}, \quad N = 1, 2, 3, \ldots \tag{57}$$

5. NUMERICAL RESULTS AND CONCLUDING REMARKS

From the preceding sections, it can be seen that the values of the stress intensity factors for the circular disc with a circumferential crack and the infinite plate weakened by a hole and a circumferential crack under any particular forms of plane loading can be obtained effectively and succinctly with accuracy through the use of corresponding equivalent procedures. In this section, a number of numerical results are presented to provide engineering practice with some useful data and to demonstrate the validity and efficiency of the equivalent procedures.

1. A disc containing a circumferential crack under uniform tension (see Table 1). This situation is equivalent to the case of

$$P(t) = -1$$

in formula (43).

2. A disc containing a circumferential crack under parabolic compression along its circular boundary (see Table 2). In this case, the disc is subject to

$$\sigma_r = -\frac{y^2}{R^2} = -\frac{(1 - \cos 2\theta)}{2}, \qquad \tau_{r\theta} = 0$$

along the circumference and acts like a rolling bearing under compression. For the sake of determining stress intensity factors, this situation is equivalent to the case of

$$P(t) = \frac{1}{2} - \frac{t^2}{4R^2} - \frac{1}{2}\left(\frac{1}{2R^2} - 1\right) t^{-2}$$

in formula (43).

Table 1. Values of stress intensity factors for a circumferentially cracked disc under uniform tension ($\overline{K}_{\rm I} = K_{\rm I}/K_{{\rm I}R\to\infty}$, $\overline{K}_{\rm II} = K_{\rm II}/K_{{\rm II}R\to\infty}$)†

	1/R								
	0.1	0.2	0.3	0.4	0,5	0.6	0.7	0.8	0.9
$\theta_0 = \pi/24$, $K_{{\rm I}R\to\infty} = 0.3590$, $K_{{\rm II}R\to\infty} = 0.02353$									
$\overline{K}_{\rm I}$	1.0053	1.0214	1.0501	1.0942	1.1591	1.2565	1.4159	1.7334	2.7513
$\overline{K}_{\rm II}$	1.0047	1.0195	1.0450	1.0811	1.1258	1.1649	1.1288	0.5941	−4.2796
$\theta_0 = \pi/12$, $K_{{\rm I}R\to\infty} = 0.4959$, $K_{{\rm II}R\to\infty} = 0.06529$									
$\overline{K}_{\rm I}$	1.0060	1.0248	1.0589	1.1139	1.2017	1.3479	1.6163	2.2079	4.1853
$\overline{K}_{\rm II}$	1.0046	1.0181	1.0397	1.0657	1.0830	1.0467	0.7925	−0.3794	−7.6122
$\theta_0 = \pi/8$, $K_{{\rm I}R\to\infty} = 0.5845$, $K_{{\rm II}R\to\infty} = 0.1163$									
$\overline{K}_{\rm I}$	1.0070	1.0294	1.0713	1.1413	1.2570	1.4551	1.8220	2.6204	5.0251
$\overline{K}_{\rm II}$	1.0034	1.0146	1.0310	1.0439	0.9544	0.9312	0.5514	−0.8676	−8.7696

† Numerical values are those of stress intensity factors at tip a (see Fig. 1).

Table 2. Values of stress intensity factors for a circumferentially cracked disc under parabolic compression†

θ_0		1/R								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\frac{\pi}{24}$	K_I	−0.3582	−0.3560	−0.3526	−0.3487	−0.3454	−0.3451	−0.3536	−0.3898	−0.5544
	10^2K_{II}	0.0159	0.0333	0.0621	0.1033	0.1623	0.2633	0.5135	1.4976	8.1162
$\frac{\pi}{12}$	K_I	−0.4954	−0.4939	−0.4923	−0.4925	−0.4981	−0.5165	−0.5654	−0.6973	−1.1811
	$10K_{II}$	0.0126	0.0176	0.0267	0.0423	0.0713	0.1348	0.3017	0.8390	3.5672
$\frac{\pi}{8}$	K_I	−0.5851	−0.5861	−0.5897	−0.5992	−0.6211	−0.6676	−0.7661	−0.9942	−1.6898
	$10K_{II}$	0.0462	0.0556	0.0746	0.1118	0.1881	0.3564	0.7610	1.9200	8.2000

† Numerical values are those of stress intensity factors at tip a (see Fig. 1).

3. A perforated infinite plate containing a circumferential crack with crack surfaces subjected to uniform compression (see Table 3). In this case, we have

$$p(t) = -1$$

in formula (55).

4. A perforated infinite plate containing a circumferential crack under uniform tension in the direction of the x-axis at infinity (see Table 4). This situation is equivalent to the case of

$$p(t) = -\frac{1}{2}(1 - R^2) + \frac{3}{2}R^2(1 - R^2)t - \frac{1}{2}(1 - R^2)t^{-2}$$

in formula (55).

5. A perforated infinite plate containing a circumferential crack under uniform tension in the direction of the y-axis at infinity (see Table 5). This situation is equivalent to the case of

$$p(t) = -\frac{1}{2}(1 - R^2) - \frac{3}{2}R^2(1 - R^2)t + \frac{1}{2}(1 - R^2)t^{-2}$$

in formula (55).

In implementing the computation in association with the numerical results contained in, for example, Table 1, we used 49 (or 80) terms of t^n in eqn (44) and 56 (or 92) terms of W in eqn (45) for $1/R$ = 0.1–0.5 and 0.6–0.9, respectively. The alternating

Table 3. Values of stress intensity factors for an infinite perforated plate containing a circumferential crack subjected to uniform compression ($\overline{K}_I = K_I/K_{IR=0}$, $\overline{K}_{II} = K_{II}/K_{IIR=0}$)†

	R								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\theta_0 = \pi/24$, $K_{IR=0} = 0.3590$, $K_{IIR=0} = 0.02353$									
$\overline{K}_I$	1.0008	1.0032	1.0077	1.0155	1.0291	1.0539	1.1054	1.2359	1.7356
$\overline{K}_{II}$	1.0014	1.0062	1.0160	1.0347	1.0722	1.1557	1.3750	2.2176	6.0805
$\theta_0 = \pi/12$, $K_{IR=0} = 0.4959$, $K_{IIR=0} = 0.06529$									
$\overline{K}_I$	1.0027	1.0112	1.0272	1.0540	1.0987	1.1758	1.3203	1.6371	2.6624
$\overline{K}_{II}$	1.0052	1.0225	1.0571	1.1218	1.2447	1.4942	2.0512	3.4881	8.4432
$\theta_0 = \pi/8$, $K_{IR=0} = 0.5845$, $K_{IIR=0} = 0.1163$									
$\overline{K}_I$	1.0049	1.0205	1.0490	1.0954	1.1688	1.2867	1.4888	1.8848	2.8532
$\overline{K}_{II}$	1.0100	1.0424	1.1059	1.2189	1.4185	1.7795	2.4637	3.8556	6.3772

† Numerical values are those of stress intensity factors at tip a (see Fig. 1).

Table 4. Values of stress intensity factors for an infinite perforated plate containing a circumferential crack under uniform tension in the direction of the x-axis at infinity†

θ_0		R								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\frac{\pi}{24}$	K_I	0.3505	0.3257	0.2862	0.2352	0.1766	0.1156	0.0581	0.0103	−0.0232‡
	K_{II}	−0.0011	−0.0040	−0.0083	−0.0136	−0.0189	−0.0230	−0.0250	−0.0239	−0.0218
$\frac{\pi}{12}$	K_I	0.4860	0.4564	0.4091	0.3470	0.2740	0.1944	0.1131	0.0347	−0.0360‡
	K_{II}	−0.0035	−0.0104	−0.0208	−0.0332	−0.0459	−0.0555	−0.0615	−0.0639	−0.0649
$\frac{\pi}{8}$	K_I	0.5757	0.5483	0.5036	0.4431	0.3684	0.2814	0.1849	0.0838	−0.0078‡
	K_{II}	−0.0077	−0.0174	−0.0321	−0.0494	−0.0671	−0.0829	−0.0955	−0.1031	−0.0915

† Numerical values are those of stress intensity factors at tip a (see Fig. 1).

‡ The alterations in the sign of K_I at $1/R = 0.9$ prove to be reasonable since in this case the σ_r near the crack tips alters itself from compressive to tensile.

Table 5. Values of stress intensity factors for an infinite perforated plate containing a circumferential crack under uniform tension in the direction of the y-axis at infinity

θ_0		R								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\frac{\pi}{24}$	K_I	0.0051	0.0200	0.0429	0.0710	0.1005	0.1266	0.1442	0.1494	0.1416
	K_{II}	0.0244	0.0267	0.0301	0.0340	0.0378	0.0405	0.0415	0.0419	0.0490
$\frac{\pi}{12}$	K_I	0.0063	0.0250	0.0545	0.0921	0.1347	0.1788	0.2208	0.2576	0.2869
	K_{II}	0.0685	0.0745	0.0836	0.0947	0.1064	0.1179	0.1298	0.1459	0.1697
$\frac{\pi}{8}$	K_I	0.0057	0.0233	0.0501	0.0825	0.1153	0.1418	0.1539	0.1419	0.0948
	K_{II}	0.1239	0.1333	0.1472	0.1627	0.1762	0.1825	0.1757	0.1487	0.0933

† Numerical values are those of stress intensity factors at tip a (see Fig. 1).

cycles needed to reach the final values in Table 1 ranged from 2 (for $1/R = 0.1$) or 3 (for $1/R = 0.2$) up to 41 or 49 (K_I or K_{II}, respectively, for $1/R = 0.9$). Using more terms involved in formulae (44) and (45) and carrying out further cycles, several cases have been computed elaborately, yet no further improvements have been obtained. This fact accounts for the validity of the computation based on the equivalent procedures.

Some significant phenomena can be observed from the numerical values in the tables. From Tables 1 and 3, it can be seen that as the distance between the crack and the circular boundary goes to zero, both K_I and K_{II} become unbounded. The screen (or protecting) effect for the crack provided by the hole can be clearly seen in Table 4 from the successive decrease of the values of K_I as the circumference approaches the crack. But the interaction between a circular boundary and a circumferential crack seems to present itself most strikingly in changing the stress field near the crack tips from essentially a single I-mode to I–II-mixed modes as the distance between them diminishes. This effect is seen from all kinds of numerical results and must have an important influence on crack propagation in media of complicated geometries.

REFERENCES

1. G. C. Sih, *Handbook of Stress-Intensity Factors*. Lehigh University, Bethlehem (1973).
2. D. P. Rooke and D. J. Cartwright, Compendium of stress intensity factors. HMSO (1976).
3. H. Tada, P. C. Paris and G. R. Irwin, *The Stress Analysis of Cracks Handbook*. Del Research Corporation (1973).
4. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*. Noordhoff, Leyden (1974).